8 Discrete Mathematics (mpf23)

(*a*) Without using the Fundamental Theorem of Arithmetic, prove that for all positive integers a, b, c, if $\gcd(b, c) = 1$ then $\gcd(a, b \cdot c) = \gcd(a, b) \cdot \gcd(a, c)$. [4 marks]

(*b*) Let k be a fixed integer.

Set $p_0 = q_0 = 1$ and, for $n \in \mathbb{N}$, let $p_{n+1} = p_n + k\, q_n$ and $q_{n+1} = p_n + q_n$.

For $n \in \mathbb{N}$, define

$$r_n = \mid k\,(q_n)^2 - (p_n)^2 \mid$$

(*i*) For $n \in \mathbb{N}$, give a closed-form expression s_n defined in terms of k and n such that $s_n = r_n$. [3 marks]

(*ii*) Prove that $s_n = r_n$ for all $n \in \mathbb{N}$. [5 marks]

(*c*) Fix sets A and B.

Consider a set P together with functions $p : P \to A$ and $q : P \to B$ such that for all sets X and for all functions $f : X \to A$ and $g : X \to B$ there exists a unique function $u\langle f, g\rangle : X \to P$ satisfying $p \circ u\langle f, g\rangle = f$ and $q \circ u\langle f, g\rangle = g$.

(*i*) Define a function from P to the product $A \times B$. [1 mark]

(*ii*) Define a function from the product $A \times B$ to P. [1 mark]

(*iii*) Prove that $u\langle p, q\rangle : P \to P$ is the identity function. [2 marks]

(*iv*) Prove that P and the product $A \times B$ are isomorphic. [4 marks]