7 Discrete Mathematics (mpf23)

(*a*) Fix positive integers m and n.

(*i*) For $k \in \mathbb{Z}$, define $[k]_m$ to be the unique element of $\mathbb{Z}_m$ congruent to k modulo m.

Prove that: $\forall\, k, \ell \in \mathbb{Z}.\, [k]_m = [\ell]_m \Leftrightarrow k \equiv \ell \pmod{m}$. [3 marks]

(*ii*) Let $f : \mathbb{Z}_m \to \mathbb{Z}_m$ be the function defined by $f(k) = [nk]_m$ and let $+_m : \mathbb{Z}_m \times \mathbb{Z}_m \to \mathbb{Z}_m$ be the function defined by $k +_m \ell = [k + \ell]_m$.

(A) Prove that: $\forall\, k, \ell \in \mathbb{Z}_m.\, f(k +_m \ell) = f(k) +_m f(\ell)$. [3 marks]

(B) Prove that f is a bijection if, and only if, $\gcd(m, n) = 1$. [6 marks]

(*b*) Recall that $\mathrm{Bij}(X, Y)$ denotes the set of bijections from a set X to a set Y and that, for $n \in \mathbb{N}$, the set $[n]$ is defined as $\{i \in \mathbb{N} \mid i < n\}$.

(*i*) Given a set A such that $0 \notin A$, describe without proof a bijection

$$\mathrm{Bij}\big(\{0\} \cup A\,,\ \{0\} \cup A\big) \to (\{0\} \cup A) \times \mathrm{Bij}(A, A)$$

[*Hint:* For $f \in \mathrm{Bij}\big(\{0\} \cup A\,,\ \{0\} \cup A\big)$ consider both $f(0)$ and $f^{-1}(0)$.]
[4 marks]

(*ii*) Using the above or otherwise, prove that: $\forall\, n \in \mathbb{N}.\ \mathrm{Bij}\big(\,[n]\,,\,[n]\,\big) \cong [n!]$.
[4 marks]