2 Advanced Algorithms (TMS)

(*a*) State the fundamental theorem of linear programming. [3 marks]

(*b*) Consider the following linear program:

$$\text{Minimize} \quad -3x_1 - 2x_2$$

subject to:

$$\begin{aligned} 3x_1 + x_2 &\leq 5 \\ -2x_1 &\geq -10 + 4x_2 \\ x_1, x_2 &\geq 0. \end{aligned}$$

(*i*) Convert this LP into standard and slack form, and specify the initial basic solution. [4 marks]

(*ii*) Solve this LP using the simplex algorithm. Specify the associated basic solution after each iteration. [4 marks]

(*c*) We consider the *Steiner Tree Problem* defined as follows. We are given an undirected, connected graph $G = (V, E)$ with a non-negative cost-function $c : E \to \mathbb{R}_+$. Further, we are given a set $S \subseteq V$ of terminals. The goal is to find a minimum-cost subgraph of G that connects all terminals, where the cost of a subgraph is the sum of the costs of its edges.

Consider the following algorithm:

- Let $H = (V, E')$ be the *metric completion* of G, where $E' = \{\{u, v\} \colon u, v \in V\}$ and $c(\{u, v\})$ is the cost of the shortest path from u to v in G.
- Compute a Minimum Spanning Tree T on the subgraph $H[S]$ induced by the set of terminals S.
- Replace every edge $\{u, v\}$ in T by the edges of a shortest path from u to v in G, and return the solution.

(*i*) Prove an upper bound of $2(1 - \frac{1}{|S|})$ on the approximation ratio of this algorithm.

[*Hint:* You can use an approach similar to the analysis of Approx-TSP-Tour.] [6 marks]

(*ii*) Construct an example which provides a matching lower bound on the approximation ratio. [3 marks]